On a new notion of the solution to an ill-posed problem *†

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Abstract

A new understanding of the notion of the stable solution to ill-posed problems is proposed. The new notion is more realistic than the old one and better fits the practical computational needs. A method for constructing stable solutions in the new sense is proposed and justified. The basic point is: in the traditional definition of the stable solution to an ill-posed problem $Au = f$, where A is a linear or nonlinear operator in a Hilbert space H, it is assumed that the noisy data $\{f_\delta, \delta\}$ are given, $||f - f_\delta|| \leq \delta$, and a stable solution $u_\delta := R_\delta f_\delta$ is defined by the relation $\lim_{\delta \to 0} ||R_\delta f_\delta - y|| = 0$, where y solves the equation $Au = f$, i.e., $Ay = f$. In this definition y and f are unknown. Any $f \in B(f_\delta, \delta)$ can be the exact data, where $B(f_\delta, \delta) := \{f : ||f - f_\delta|| \leq \delta\}$.

The new notion of the stable solution excludes the unknown y and f from the definition of the solution.

1 Introduction

Let

$$Au = f, \tag{1.1}$$

where $A : H \to H$ is a linear closed operator, densely defined in a Hilbert space H. Problem (1.1) is called ill-posed if A is not a homeomorphism of H onto H, that is, either equation (1.1) does not have a solution, or the solution is non-unique, or the solution does not depend on f continuously. Let us assume that (1.1) has a solution, possibly non-unique. Let $N(A)$ be the null space of A, and y be the unique normal solution to

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(1.1), i.e., $y \perp N(A)$. Given noisy data f_δ, $\|f_\delta - f\| \le \delta$, one wants to construct a stable approximation $u_\delta := R_\delta f_\delta$ of the solution y, $\|u_\delta - y\| \to 0$ as $\delta \to 0$.

Traditionally (see, e.g., [2]) one calls a family of operators R_h a regularizer for problem (1.1) (with not necessarily linear operator A) if
a) $R_h A(u) \to u$ as $h \to 0$ for any $u \in D(A)$,
b) $R_h f_\delta$ is defined for any $f_\delta \in H$ and there exists $h = h(\delta) \to 0$ as $\delta \to 0$ such that

$$\|R_{h(\delta)} f_\delta - y\| \to 0 \text{ as } \delta \to 0. \tag{$*$}$$

In this definition y is fixed and $(*)$ must hold for any $f_\delta \in B(f, \delta) := \{f_\delta : \|f_\delta - f\| \le \delta\}$.

In practice one does not know the solution y and the exact data f. The only available information is a family f_δ and some *a priori information about f or about the solution y*. This a priori information often consists of the knowledge that $y \in \mathcal{K}$, where $\mathcal{K}$ is a compactum in H. Thus

$$y \in S_\delta := \{v : \|A(v) - f_\delta\| \le \delta,\ v \in \mathcal{K}\}.$$

We assume that the operator A is known exactly, and we always assume that $f_\delta \in B(f, \delta)$, where $f = A(y)$.

Definition: *We call a family of operators $R(\delta)$ a regularizer if*

$$\sup_{v \in S_\delta} \|R(\delta) f_\delta - v\| \le \eta(\delta) \to 0 \quad \text{as} \quad \delta \to 0. \tag{1.2}$$

There is a *crucial difference* between our new Definition (1.2) and the standard definition $(*)$:

In $(*)$ *u is fixed, while in* (1.2) *v is an arbitrary element of S_δ and the supremum of the norm in* (1.2) *over all such v must tend to zero as $\delta \to 0$.*

The new definition is more realistic and better fits computational needs because not only the solution y to (1.1) satisfies the inequality $\|Ay - f_\delta\| \le \delta$, but any $v \in S_\delta$ satisfies this inequality $\|Av - f_\delta\| \le \delta$, $v \in \mathcal{K}$. The data f_δ may correspond to any $f = Av$, where $v \in S_\delta$, and not only to $f = Ay$, where y is a solution of equation (1.1). Therefore it is more natural to use definition (1.2) than $(*)$.

Our goal is to illustrate the practical difference between these two definitions, and to construct regularizer in the sense (1.2) for problem (1.1) with an arbitrary, not necessarily bounded, linear operator A, which is closed and densely defined in H. This is done in Section 2. In Section 1 this is done for a class of equations (1.1) with nonlinear operators $A : X \to Y$, where X and Y are Banach spaces. In this case we assume that
A1) *$A : X \to Y$ is a closed, nonlinear, injective map,* $f \in \mathcal{R}(A)$, $\mathcal{R}(A)$ it is the range of A,

and
A2) $\phi : D(\phi) \to [0, \infty)$, $\phi(u) > 0$ *if* $u \ne 0$, $D(\phi) \subseteq D(A)$, *the sets* $\mathcal{K} = \mathcal{K}_c := \{v : \phi(v) \le c\}$ *are compact in* X *for every* $c = const > 0$, *and if* $v_n \to v$, *then* $\phi(v) \le \liminf_{n\to\infty} \phi(v_n)$.

The last inequality holds if ϕ is *lower semicontinuous.* In Hilbert spaces and in reflexive Banach spaces norms are lower semicontinuous.

Let us give some examples of equations for which assumptions A1) and A2) are satisfied.

Example 1. A is a linear injective compact operator, $f \in \mathcal{R}(A)$, $\phi(v)$ is a norm on $X_1 \subset X$, where X_1 is densely imbedded in X, the embedding $i : X_1 \to X$ is compact, and $\phi(v)$ is lower semicontinuous.

Example 2. A is a nonlinear injective continuous operator $f \in \mathcal{R}(A)$, A^{-1} is not continuous, ϕ is as in Example 1.

Example 3. A is linear, injective, densely defined, closed operator, $f \in \mathcal{R}(A)$, A^{-1} is unbounded, ϕ is as in Example 1, $X_1 \subseteq D(A)$.

Let us demonstrate by Example A that *a regularizer in the sense* $(*)$ *may be not a regularizer in the sense* (1.2).

In Example B a theoretical construction of a regularizer in the sense (1.2) is given for some equations (1.1) with nonlinear operators.

In Section 2 a novel theoretical construction of a regularizer in the sense (1.2) is given for a very wide class of equations (1.1) with linear operators A.

Example A: *Stable numerical differentiation.*

In this Example the results from [3] - [11] are used. This Example is borrowed from [10].

Consider stable numerical differentiation of noisy data. The problem is:

$$Au := \int_0^x u(s)\,ds = f(x), \quad f(0) = 0, \quad 0 \le x \le 1. \tag{1.3}$$

The data are: f_δ and a constant M_a, which defines a compact $\mathcal{K}$, where $\|f_\delta - f\| \le \delta$, the norm is $L^\infty(0,1)$ norm, and $\mathcal{K}$ consists of the L^∞ functions which satisfy the inequality $\|u\|_a \le M_a$, $a \ge 0$. The norm

$$\|u\|_a := \sup_{\substack{x,y\in[0,1]\\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^a} + \sup_{0\le x\le 1} |u(x)| \quad \text{if } 0 \le a \le 1,$$

$$\|u\|_a := \sup_{0\le x\le 1} (|u(x)| + |u'(x)|) + \sup_{\substack{x,y\in[0,1]\\ x\ne y}} \frac{|u'(x)-u'(y)|}{|x-y|^{a-1}}, \quad 1 < a \le 2.$$

If $a > 1$, then we define

$$R(\delta) f_\delta := \begin{cases} \frac{f_\delta(x+h(\delta)) - f_\delta(x-h(\delta))}{2h(\delta)}, & h(\delta) \le x \le 1 - h(\delta), \\ \frac{f_\delta(x+h(\delta)) - f_\delta(x)}{h(\delta)}, & 0 \le x < h(\delta), \\ \frac{f_\delta(x) - f_\delta(x-h(\delta))}{h(\delta)}, & 1 - h(\delta) < x \le 1, \end{cases} \tag{1.4}$$

where

$$h(\delta) = c_a \delta^{\frac{1}{a}}, \tag{1.5}$$

and c_a is a constant given explicitly (cf [4]).

We prove that (1.4) is a regularizer for (1.3) in the sense (1.2), and $\mathcal{K} := \{v : \|v\|_a \leq M_a,\ a > 1\}$. In this example we do not use lower semicontinuity of the norm $\phi(v)$ and do not define ϕ.

Let $S_{\delta,a} := \{v : \|Av - f_\delta\| \leq \delta,\ \|v\|_a \leq M_a\}$. To prove that (1.4)-(1.5) is a regularizer in the sense (1.2) we use the estimate

$$\begin{aligned}\sup_{v\in S_{\delta,a}} \|R(\delta)f_\delta - v\| \leq \sup_{v\in S_{\delta,a}} \{\|R(\delta)(f_\delta - Av)\| + \|R(\delta)Av - v\|\} \leq \frac{\delta}{h(\delta)} + M_a h^{a-1}(\delta) \leq \\ \leq c_a \delta^{1-\frac{1}{a}} := \eta(\delta) \to 0 \text{ as } \delta \to 0.\end{aligned} \tag{1.6}$$

Thus we have proved that (1.4)-(1.5) is a regularizer in the sense (1.2).

If $a = 1$, and $M_1 < \infty$, then one can prove the following result:

Claim: *There is no regularizer for problem* (1.3) *in the sense* (1.2) *even if the regularizer is sought in the set of all operators, including nonlinear ones.*

More precisely, it is proved in [5], p.345, (see also [8], pp 197-235, where the stable numerical differentiation problem is discussed in detail) that

$$\inf_{R(\delta)} \sup_{v\in S_{\delta,1}} \|R(\delta)f_\delta - v\| \geq c > 0,$$

where $c > 0$ is a constant independent of δ and the infimum is taken *over all operators* $R(\delta)$ *acting from* $L^\infty(0,1)$ *into* $L^\infty(0,1)$*, including nonlinear ones.*

On the other hand, if $a = 1$ and $M_1 < \infty$, then a regularizer in the sense $(*)$ does exist, but the rate of convergence in (*) may be as slow as one wishes, if $u(x)$ is chosen suitably (see [4], [8]).

Example B: *Construction of a regularizer in the sense* (1.2) *for some nonlinear equations.*

Assuming A1) and A2), let us construct a regularizer for (1.1) in the sense (1.2). We use the ideas from [10] and [11].

Define $F_\delta(v) := \|Av - f_\delta\| + \delta\phi(v)$ and consider the minimization problem of finding the infimum $m(\delta)$ of the functional $F_\delta(v)$ on a set S_δ:

$$m(\delta) := \inf_{v\in S_\delta} F_\delta(v), \qquad S_\delta := \{v : \|Av - f_\delta\| \leq \delta,\ \phi(v) \leq c\}. \tag{1.7}$$

Here

$$\mathcal{K} = \mathcal{K}_c := \{v : \phi(v) \leq c\}.$$

The constant $c > 0$ can be chosen arbitrary large and fixed at the beginning of the argument, and then one can choose a smaller constant c_1, specified below. Since $F_\delta(u) = \delta + \delta\phi(u) := c_1\delta$, $c_1 := 1 + \phi(u)$, where u solves (1.1), one concludes that

$$m(\delta) \leq c_1\delta. \tag{1.8}$$

Let v_j be a minimizing sequence and $F_\delta(v_j) \leq 2m(\delta)$. Then $\phi(v_j) \leq 2c_1$. By assumption A2), as $j \to \infty$, one has:

$$v_j \to v_\delta, \qquad \phi(v_\delta) \leq 2c_1. \tag{1.9}$$

Take $\delta = \delta_m \to 0$ and denote $v_{\delta_m} := w_m$. Then (1.9) and Assumption A2) imply the existence of a subsequence, denoted again w_m, such that:

$$w_m \to w, \qquad A(w_m) \to A(w), \qquad \|A(w) - g\| = 0. \tag{1.10}$$

Thus $A(w) = g$ and, since A is injective by Assumption A1), it follows that $w = u$, where u is the unique solution to (1.1).

Define now $R(\delta)f_\delta$ by the formula $R(\delta)f_\delta := v_\delta$, where v_δ is defined in (1.9).

Theorem 1.1. *$R(\delta)$ is a regularizer for problem (1.1) in the sense (1.2).*

Proof. Assume the contrary:

$$\sup_{v\in S_\delta} \|R(\delta)f_\delta - v\| = \sup_{v\in S_\delta} \|v_\delta - v\| \geq \gamma > 0, \tag{1.11}$$

where $\gamma > 0$ is a constant independent of δ. Since $\phi(v_\delta) \leq 2c_1$ by (1.9), and $\phi(v) \leq c$, one can choose convergent in X sequences $w_m := v_{\delta_m} \to \tilde{w}$, $\delta_m \to 0$, and $v_m \to \tilde{v}$, such that $\|w_m - v_m\| \geq \frac{\gamma}{2}$, $\|\tilde{w} - \tilde{v}\| \geq \frac{\gamma}{2}$, and $A(\tilde{w}) = g$, $A(\tilde{v}) = g$. By the injectivity of A it follows that $\tilde{w} = \tilde{v} = u$. This contradicts the inequality $\|\tilde{w} - \tilde{v}\| \geq \frac{\gamma}{2} > 0$. This contradiction proves the theorem.

The conclusions $A(\tilde{w}) = g$ and $A(\tilde{v}) = g$, that we have used above, follow from the inequalities $\|A(v_\delta) - f_\delta\| \leq \delta$ and $\|A(v) - f_\delta\| \leq \delta$ after passing to the limit $\delta \to 0$, using assumption A2). $\square$

2 Construction of a regularizer in the sense (1.2) for linear equations

If A is a linear closed densely defined in H operator, then $T = A^*A$ is a densely defined selfadjoint operator. Let $T_a := T + aI$, where $a = \mathit{const} > 0$. The operator $T_a^{-1}A^*$ is densely defined and closable. Its closure is a bounded operator, defined on all of H, and $||T_a^{-1}A^*|| \leq \frac{1}{2\sqrt{a}}$. See [12]-[15] for details and other results. Let E_s be the resolution of the identity of the selfadjoint operator T, $d\rho := d(E_s y, y)$, and $\mathcal{K} := \{u : \int_0^\infty s^{-2p} d\rho \leq k_p^2\}$, where $p \in (0, 1)$ and $k_p > 0$ are constants.

Our basic result is:

Theorem 2.1. *The operator $R_\delta = T_{a(\delta)}^{-1}A^*$ is a regularizer for problem (1.1) in the sense (1.2) if $\lim_{\delta\to 0} \frac{\delta}{a(\delta)^{1/2}} = 0$ and $\lim_{\delta\to 0} a(\delta) = 0$. Moreover, if $a(\delta) = b_p\delta^{\frac{2}{2p+1}}$, then*

$$\sup_{y\in\mathcal{K}, ||Ay - f_\delta|| \leq \delta} \|R(\delta)f_\delta - y\| \leq C_p\delta^{\frac{2p}{2p+1}}, \tag{2.1}$$

where

$$C_p = \frac{1}{2\sqrt{b_p}} + c_p k_p b_p^p, \qquad c_p = p^p(1-p)^{1-p}, \qquad b_p := (4pc_pk_p)^{-\frac{2}{2p+1}}.$$

The above choice of $a(\delta)$ is optimal in the sense that the right-hand side of (2.2) *(see below) is minimal for this choice of $a(\delta)$.*

Proof.

Let

$$\epsilon := \sup_{y\in\mathcal{K}, ||Ay-f_\delta||\le\delta} ||T_a^{-1}A^*f_\delta - y|| := \sup ||T_a^{-1}A^*f_\delta - y||.$$

Then, with $Ay = f$, one has

$$\epsilon \le \sup ||T_a^{-1}A^*(f_\delta - f)|| + \sup ||T_a^{-1}A^*Ay - y|| := J_1 + J_2,$$

where

$$J_1 \le \frac{\delta}{2\sqrt{a}},$$

and

$$J_2^2 \le \sup\{a^2||T_a^{-1}y||^2\} \le \sup \int_0^\infty \frac{a^2}{(s+a)^2} d(E_s y, y).$$

Thus,

$$J_2^2 \le \big(\max_{s\ge 0} \frac{as^p}{a+s}\big)^2 k_p^2 = c_p^2 k_p^2 a^{2p},$$

because $\max_{s\ge0} \frac{as^p}{a+s}$ is attained at $s = \frac{pa}{1-p}$ and is equal to $c_p a^p$, where

$$c_p := p^p(1-p)^{1-p}, \qquad k_p^2 := \sup_{y\in\mathcal{K}} \int_0^\infty s^{-2p} d(E_s y, y).$$

Consequently,

$$J_2 \le c_p k_p a^p,$$

and

$$\epsilon \le \frac{\delta}{2\sqrt{a}} + c_p k_p a^p. \tag{2.2}$$

Minimizing the right-hand side of (2.2) with respect to $a > 0$, one obtains inequality (2.1).

The minimizer of the right-hand side of (2.2) is

$$a = a(\delta) = b_p \delta^{\frac{2}{2p+1}}, \qquad b_p := (4pc_p k_p)^{-\frac{2}{2p+1}},$$

and the minimum of the right-hand side of (2.2) is $C_p \delta^{\frac{2p}{2p+1}}$, where

$$C_p := \frac{1}{2\sqrt{b_p}} + c_p k_p b_p^p. \tag{2.3}$$

Theorem 2.1 is proved. $\square$

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